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Tensor conditions for algebraic spinors

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Abstract. The spinor may be defined as an element of a minimal ideal (primitive element) in a Clifford algebra. We pose the question: what are the necessary and sufficient covariant conditions to be imposed on the antisymmetric tensors of a Clifford number such that it is primitive? The solution to this problem is illustrated for the case of the real Pauli algebra. Special cases of the conditions yield the non-relativistic two-component spinors in spin space and spinor–tensor relations given by previous writers.

1. Introduction

It has long been known that, instead of regarding the spinor as simply an operand in a representation space (a 'column'), it may be considered as embedded within the Clifford algebra that operates on it. This idea originated with Sauter (1930) who showed how the Dirac equation could be solved with the aid of algebraic techniques without recourse to matrix representations for the Dirac operators, a method later extended by Kähler (1961, 1962) and more recently by Greider (1980). The precise nature of the Clifford-algebraic substructure of which the spinor is a part—a minimal ideal—was clarified by Riesz (1946, 1953). Thus the use of a full geometric algebra, in which one can add tensors of different rank, incorporates within one formalism the spin spaces of the group-theoretical approach (minimal ideals), up and down spinor indices (left and right ideals) and the invariance groups. In this way one obtains a conception of spinor theory of great generality and unity which a number of physicists have studied in various contexts (e.g. Coquereaux 1982, Doria 1977, Graf 1978, Hestenes 1966, 1975, Teitler 1965). This view of things has not been widely exploited, however.

One problem in the algebraic theory that has not been adequately explored relates to the tensor representation of spinors. Several authors have attempted to clarify the meaning and significance of the column spinor by relating bilinear combinations of its components to local tensorial structures located in space (Euclidean three-space E_3 or space–time) (see for example Cartan 1966, Kramers 1957, Penrose 1968 and Takabayasi 1955, 1957). A comparable approach to the tensorisation of spinor theory which starts from the algebraic definition has not been well developed. In fact this problem arises naturally when one realises that, in the algebraic approach, a spinor is just a special kind of Clifford number ('primitive', i.e. one which generates and lies in a minimal ideal (left or right)), and that a Clifford number is a superposition of multidimensional exterior forms with antisymmetric tensor coefficients. Thus it might be expected that a necessary and sufficient algebraic condition could be imposed on a Clifford number such that it is primitive, with equivalent covariant subsidiary relations among the aggregate of forms. This would yield the fullest possible characterisation of the spinor, in terms of tensors, implied and allowed by the algebraic definition. A treatment of this kind is, indeed, possible although to the author's knowledge this problem has not been clearly posed in the literature.

To be consistent such a procedure must, of course, reproduce the known spinor–tensor relations. It has proved possible in prior work to interpret most of the facets of the abstract quantum formalism in terms of, for example, vectors and bivectors and their interrelationships, apart from an ambiguity in the sign of the spinor. However, the motivation for the tensor constructions used in the literature, in particular for the spinors of E_3, has not always been clear. In our approach, this problem of representation is treated systematically and falls into two parts. The initial task is to discover the conditions under which a Clifford element is primitive. In § 2 we discuss this problem in general terms and suggest two methods of solution. One of these can, in principle, be applied to any Clifford algebra; the other method, although more elegant, applies only to certain kinds of Clifford algebra. Once the nature of the primitivity restriction is established, a discussion of the relations between the spinor and the algebraic notion of spin space on the one hand, and tensors on the other hand, can be entered into.

In this paper we shall only be concerned with carrying through the above procedure for the case of the real Pauli algebra C_3 (the universal Clifford algebra of E_3 where the tensor coefficients are real). Both methods apply here. In § 3 various forms of the primitivity condition are given and the equivalent tensor relations stated. As a by-product of the main algebraic result the general form of elements which are of particular importance to the structure of the algebra may be deduced; for example we give all the primitive idempotents in C_3.

It is when a matrix representation is chosen for the basis elements of the algebra that we make contact with the column spinors of the group-theoretical approach (in the present case the two-component non-relativistic spinors). In fact, the requirement of primitivity is the minimum necessary for a Clifford element to lie in a minimal ideal and we find for C_3 that this is related to a product of two distinct dual column (Pauli) spinors. Restrictive assumptions must be made on the form of a primitive element, with equivalent restrictions on the representative matrix, in order to recover the element as part of a spin space or to derive the spinor–tensor relations found by previous writers. In the former case restricting the primitivity condition in a non-covariant way yields the linear components of a Pauli spinor as coefficients in a minimal ideal and, in the latter case, restrictions of a covariant nature imply various relations between tensors in E_3 and quadratic combinations of the components of a single Pauli spinor. These matters are treated in §§ 4 and 5.

As has often been noted, the physical observables defined by the bilinear combinations are not sensitive to the reversal of sign of the spinor wavefunction on rotation in space, although this property gives rise to real physical effects, as has been experimentally verified (Rauch *et al* 1975, Werner *et al* 1975). The sign reversal manifests itself in space in a non-local way (Bohm and Schiller 1956, Misner *et al* 1973, Penrose 1968), for example through the relative orientation of an object with its surroundings, and cannot be made manifest within the context of a purely local tensor theory of spinors. Our results confirm this and in the present paper we shall deal only with such a local approach. Also, our considerations are purely algebraic in that restrictions imposed by a wave equation, or indeed any relation involving derivatives, are not considered.

We should mention here that this work is closely related to the algebraic E-number theory of Eddington (1946). Certain preferred elements in Eddington's theory have the property of being 'pure', that is the matrices representing the elements factorise into a column times a row. In a representation (*any* representation) this is indeed the content of our primitivity condition. However, our approach follows a different route to that of Eddington who neither derived his condition, nor related it in its algebraic form to equivalent tensor conditions, in the ways we are doing.

The detailed physical interpretation of the results of this paper, and their extension to the Dirac equation, will be discussed elsewhere. The motivation for the work derives from a wider study into the topological and algebraic basis of physics which makes use of de Rham cohomology theory as a means of bringing a common algebraic topological language to physical laws. This approach involves reformulating laws in the language of differential forms and then reinterpreting the content of the laws by writing them as topological statements. The relation between spinors and forms as presented here, and the algebraic structure that underlies that relation, is clearly of relevance to such a programme (a full discussion of this, which embraces electromagnetism and gravitation, is given by Holland (1981)).

2. Primitive elements and algebraic spinors

In this section we shall state two theorems, one of which is quoted, the other proved, which are basic to our approach, both being concerned with the conditions for primitivity. The results relate to associative algebras in general and do not rely on an algebra being specifically a Clifford algebra. We then describe how these theorems may be used in the context of a Clifford algebra to carry through the programme outlined in § 1. Details of the mathematical background assumed below may be found in the works of Abian (1971), Albert (1961) and Weyl (1950).

Recall that a left ideal I in an algebra A (which we suppose has a unity element 1) is a subalgebra such that for $c \in I$ and any $a \in A$, $ac \in I$, and that I is minimal if it contains no left ideal other than itself or zero.

Choose within an algebra A a primitive idempotent E. Our reason for concentrating on such an element is that the only known theorems which are of use to us concerning elements which generate minimal ideals relate to idempotents. An example is given by Albert (1961, pp 26 and 40) as follows.

Theorem 1.

(i) Suppose E is an idempotent in an algebra A. Then E is primitive if and only if E is the only idempotent of EAE. Further:

(ii) If A is simple, E is primitive if and only if $EAE = DE$, i.e.

$$ECE = zE \text{ for all } C \in A, \text{ some } z \in D \tag{2.1}$$

where D is a division algebra appearing in the Wedderburn decomposition of A whose unity element 1 coincides with that of A. The unity element in the division algebra DE is E and $Ez = zE$ for all $z \in D$.

We now give the basic condition for any element in an algebra A to be primitive. Consider a minimal left ideal which has a primitive idempotent E as its unity element so that any element of the form ψE, $\psi \in A$, lies in the ideal and, indeed, generates it.

Then ψ itself is an element of a minimal left ideal if and only if

$$\psi = \psi E. \tag{2.2}$$

Similarly ψ is an element of a minimal right ideal if and only if $\psi = E\psi$. Either of these conditions characterise ψ as primitive (i.e., ψ factorises into itself pre- or post-multiplied by some E). When A is a Clifford algebra such ψ's are called 'algebraic spinors' or more particularly left and right algebraic spinors, respectively.

In the event that A is simple and that the centre $Z \subseteq A$ and the division algebra D coincide, we can give the criterion for the primitivity of ψ in a form that does not depend on a particular generating idempotent. Obviously, not all Clifford algebras possess these properties, but the real Pauli algebra (see § 4) and the Dirac algebra do.

Theorem 2.
$\psi = \psi E$, where E satisfies the provisions of theorem 1(ii) with $D = Z$, if and only if

$$\psi C \psi = \psi z \text{ for all } C \in A \text{ and some } z \in Z. \tag{2.3}$$

For, from theorem 1, for any $C \in A$, $\psi C \psi = \psi E(C\psi)E = \psi E z = \psi z$. Conversely, $\psi C \psi = \psi z$ implies $(\psi z^{-1})C(\psi z^{-1}) = \psi z^{-1}$ for all $C \in A$ so that, putting $C = 1$, there is a $z' \in Z$, for which $E = \psi z'^{-1}$ is idempotent, and $ECE = z'^{-1} z E$ for all C, whence, by theorem 1, E is primitive. Thus, there exists a primitive idempotent $E = \psi z'^{-1}$ with $\psi = \psi E$. This theorem holds equally well, of course, for a right spinor.

Given the stated restrictions on A, (2.3) gives the minimum condition on ψ for it to factorise in the form (2.2) without depending in any way on the specific form of E. Naturally, ψ may satisfy conditions additional to (2.3) when a particular form for E is chosen in (2.2).

What we require is to go further and state the primitivity condition purely in terms of ψ, without reference to any other algebraic element, when A is a Clifford algebra. Two methods present themselves. Firstly, when the requirements of theorem 2 are satisfied, this can be accomplished by letting ψ and C be general Clifford numbers and finding the restriction on ψ such that (2.3) is obeyed for all C. Although this yields the condition in a straightforward way, it involves somewhat longwinded calculations. Secondly, an easier method is simply to take a particular primitive idempotent E and solve (2.2) as a set of simultaneous equations by eliminating the idempotent unknowns to leave conditions on ψ. However, in so doing we must be careful not to import into the result any restriction on ψ which depends on the specific form of the E chosen. We can ensure this by using a covariant (in the sense of coordinate transformations) form of idempotent in which basis elements of the algebra are not singled out in a preferred manner. The spirit of this approach is: given one primitive element E, we can find all primitive elements. The second method has the advantage of being applicable to any Clifford algebra. These points will now be clarified through the study of an example.

3. Tensor conditions for the algebraic Pauli spinor

3.1. The real Pauli algebra

We shall find in this section the algebraic and tensorial conditions for an aggregate in the real Pauli algebra C_3 to be primitive. A general number $C \in C_3$ is a superposition of scalar S, vector V, bivector B and pseudoscalar P terms:

$$C = S + V + B + P \tag{3.1}$$

where $V = V_i e^i$, $B = \frac{1}{2} B_{ij} e^{ij}$, $P = (1/3!) P_{ijk} e^{ikj} = P_{123} e^{123}$, $i, j, k = 1, 2, 3$, C_3 being a linear associative algebra generated by 1 and the one-forms e^i, $i = 1, 2, 3$ with inner product

$$e^i \cdot e^j = \tfrac{1}{2}(e^i e^j + e^j e^i) = g^{ij} \equiv \delta^{ij} \tag{3.2}$$

(we need not distinguish between forms and vectors). We have used the notation $e^{i_1 \ldots i_r} \equiv e^{i_1} \wedge \ldots \wedge e^{i_r}$ for the outer product. The tensor coefficients are real; it should be noted that when C_3 is represented over the complex numbers, the non-relativistic two-component spinors with complex components are recovered from the algebraic spinors (§ 4). The centre Z comprises scalar and pseudoscalar elements.

We now give some useful formulae (which are valid for any Clifford algebra). The inner product of an r-form R and a q-form Q, with $q \geqslant r$, is a $(q-r)$-form $R \cdot Q$ given by

$$R \cdot Q = \frac{(-1)^{r(r-1)/2}}{(q-r)!\,r!\,q!} \delta^{j_1 \ldots j_q}_{i_1 \ldots i_r k_1 \ldots k_{q-r}} R^{i_1 \ldots i_r} Q_{j_1 \ldots j_q} e^{k_1 \ldots k_{q-r}}$$

with tensor components

$$(R \cdot Q)_{k_1 \ldots k_{q-r}} = [(-1)^{\frac{1}{2}r(r-1)}/r!] R^{i_1 \ldots i_r} Q_{i_1 \ldots i_r k_1 \ldots k_{q-r}} \tag{3.3}$$

where the indices on R have been raised using g^{ij}. The components of the outer product are

$$(R \wedge Q)_{k_1 \ldots k_{r+q}} = (1/r!)(1/q!) \delta^{i_1 \ldots i_r j_1 \ldots j_q}_{k_1 \ldots k_{r+q}} R_{i_1 \ldots i_r} Q_{j_1 \ldots j_q}. \tag{3.4}$$

The Clifford product of an r-form and a q-form contains, in general, intermediate terms other than the extremes of the $(q-p)$ inner product and the $(q+p)$ outer product. Since the only term not to depend on metric in such a product is the outer product, we call the intermediate terms 'semi-inner products'. In the case of the Pauli algebra the only semi-inner product to occur in a product of two Pauli numbers is a bivector (two-form) term (B_1, B_2) formed from two bivectors B_1, B_2. This may be rewritten in terms of the inner and outer products of the factors as follows

$$(B_1, B_2) = {}^*B_2 \wedge {}^*B_1 = {}^*(B_1 \cdot {}^*B_2)$$

with the Hodge dual ${}^*B = \frac{1}{2}\varepsilon_{ijk} B^{jk} e^i$. In particular $(B, B) = 0$. All calculations within C_3 are therefore expressible in terms of just the inner and outer products of forms. With the help of the above formulae it is straightforward to prove a series of identities, for example

$$(V \cdot B_1) \cdot B_2 = V(B_1 \cdot B_2) - B_1 \cdot (V \wedge B_2) \qquad (B_1 \cdot B_2) \cdot P = B_1 \cdot (P \cdot B_2) \text{ etc}$$

although we shall not list them here. These have been used without comment below.

One further result will be of use. Apply successively to C the invariant operations of reversing the directions of all one-forms ($\bar{C} = S - V + B - P$: the space conjugate) and reversing the order of products of one-forms ($\bar{\bar{C}} = S + V - B - P$: the Hermitian conjugate), which operations commute. The result is $\tilde{C} = S - V - B + P$. Finally, form the product

$$C\bar{\bar{\tilde{C}}} = (S^2 - V \cdot V - B \cdot B + P \cdot P) + 2(SP - V \wedge B). \tag{3.5}$$

Further details on Clifford algebras may be found in Hestenes (1966), Kähler (1960), Porteous (1969) and Riesz (1958).

3.2. *Condition for primitivity*

3.2.1. First method. The assumptions of theorem 2 hold for C_3 and so the condition for $\psi \in C_3$ to be primitive is given by (2.3). For ψ of the form (3.1) and an arbitrary $c = s + v + b + p$ it may be shown that, quite generally,

$$\psi c \psi = z\psi - \psi\tilde{\bar{\psi}}\tilde{c} \tag{3.6}$$

where we have used (3.5) and $z = 2[(sS + v \cdot V + b \cdot B + p \cdot P) + (sP + v \wedge B + b \wedge V + pS)] \in Z$. It is straightforward to prove from (3.6) that $\psi c \psi = z\psi$ for all $c \in C_3$ and some $z \in Z$ if and only if

$$\psi\tilde{\bar{\psi}} = 0. \tag{3.7}$$

This solves the problem of stating the condition for ψ to be primitive purely in terms of ψ. (3.7) is equivalent to

$$S^2 - V \cdot V - B \cdot B + P \cdot P = 0 \qquad SP = V \wedge B \tag{3.8}$$

and since

$$\psi^2 = (S^2 + V \cdot V + B \cdot B + P \cdot P) + 2(SV + B \cdot P) + 2(SB + V \cdot P) + 2(SP + V \wedge B) \tag{3.9}$$

to

$$\psi^2 = 2(S + P)\psi \tag{3.10}$$

which gives the condition in the form closest to (2.3).

Using (3.3) and (3.4) the alternative forms of the condition are summarised in table 1. There is a degeneracy in these conditions in that, if ψ satisfies them, then so do $\bar{\psi}$, $\tilde{\psi}$, $\tilde{\bar{\psi}}$ and $s\psi$ (s a scalar). In addition, the unit pseudoscalar, when applied to a q-form Q, is related to the Hodge duality operator, the precise relation being

$$e^{123}Q = (-1)^{\frac{1}{2}q(q-1)} {*Q} \tag{3.11}$$

with, since $** = (-1)^{q(3-q)}$, $(e^{123})^2 = -1$ as required. If we define the Hodge dual of

Table 1. Five equivalent necessary and sufficient conditions for a real Pauli number $\psi = S + V + B + P \in C_3$ to be primitive. These conditions leave six degrees of freedom in ψ arbitrary.

(1) Coordinate free	$S^2 - V \cdot V - B \cdot B + P \cdot P = 0$ $SP = V \wedge B$
(2) Components $i, j, k = 1, 2, 3$	$S^2 - V_iV^i + \frac{1}{2}B_{ij}B^{ij} - P_{123}^2 = 0$ $SP_{123} = \frac{1}{2}\varepsilon^{ijk}V_iB_{jk}$
(3) Algebraic	$\psi\tilde{\bar{\psi}} = \tilde{\bar{\psi}}\psi = 0$
(4) Algebraic-primitive	$\psi^2 = 2(S + P)\psi$
(5) Matrix	$\psi^2 = (\mathrm{Tr}\,\psi)\psi$ $\psi^a_b = \xi^a\eta_b$

a Pauli number ψ to be $^*\psi = {}^*S + {}^*V + {}^*B + {}^*P$ then it is true that $^*\psi$ also satisfies the conditions of table 1.

3.2.2. Second method. We now show how to recover the table 1 conditions directly from (2.2) without recourse to (2.3). First of all we require a primitive idempotent E—in solving (2.2) for ψ it is not necessary to use the most general primitive idempotent in C_3 (which is an element constrained by $E = E^2$ and theorem 1). Since C_3 is simple, theorem 1(ii) is applicable and it is readily verified that the idempotent

$$E = \tfrac{1}{2}(1+u) \tag{3.12}$$

where u is a unit vector, is primitive, since for any C as in (3.1), $ECE = zE$ where $z = s + p \in Z$ with $s = S + u \cdot V$, $p = u \wedge B + P$ (and $z^{-1} = (s^2 - p \cdot p)^{-1}(s-p)$).

The idempotent $E' = \frac{1}{2}(1-u)$ is also primitive and so C_3 may be decomposed into the direct sum of two minimal left ideals:

$$C_3 = C_3E \oplus C_3E' \qquad EE' = E'E = 0 \tag{3.13}$$

i.e. any real Pauli element is a sum of two primitive elements.

Consider then the left spinor (2.2) which lies in the ideal generated by (3.12):

$$\psi = \psi u. \tag{3.14}$$

We could fix E further by choosing $u = e^3$, say. For ψ of the form (3.1), (3.14) then yields

$$S = V_3 \qquad V_1 = B_{13} \qquad V_2 = B_{23} \qquad B_{12} = P_{123}. \tag{3.15}$$

This non-covariant restriction has no particular significance and bears out the remark in § 2 that we must not single out in E the basis elements in terms of which the multiforms of ψ are expressed if we are to discover a covariant condition on ψ. Maintaining a general u, (3.14) gives eight equations of which only four are independent. Since $u \cdot u = 1$ the elimination of u from (3.14) will leave at most two conditions for the tensors of ψ. We omit the details but it is easy to show from (3.3) and (3.4) that this procedure yields (3.8) and $su = (V \cdot V - P \cdot P)V - S(B \cdot V) - B \cdot (V \cdot B)$ with $s = S(V \cdot V - P \cdot P)$. Now, conversely to this derivation, starting with the relations (3.8) and substituting u as just given into the right-hand side of (3.14), we recover the left-hand side of this latter set of equations. It follows that (3.8) are the necessary and sufficient (covariant) conditions for $\psi \in C_3$ to factorise in the form (3.14) and so be an algebraic Pauli left spinor.

Similarly, the constraints (3.8) are consequent to, and imply, the condition for ψ to be an algebraic right spinor, that is $\psi = E\psi$ with E of the form (3.12)†. Evidently, although this method operates via the use of a particular choice of primitive idempotent, we are able to deduce the general condition for ψ to generate a minimal ideal (so that ψ factorises in the form (2.2) where E is *any* primitive idempotent).

3.3. Particular cases (table 2).

We can enumerate all the primitive idempotents in the real Pauli algebra from (3.10) by noting that, in addition to being primitive, ψ is idempotent if and only if $\psi^2 = 2(S+P)\psi = \psi$. Writing out the second equality shows that if P is real then $P = 0$

† Where of course u now has a different form in terms of ψ. In fact, $\psi = \psi u$ is equivalent to $\tilde{\psi} = u\tilde{\psi}$, same u.

Table 2. Special types of primitive elements in C_3 with equivalent matrix restrictions additional to $\psi = \xi\eta$.

Idempotent	$S=\frac{1}{2}$, $V\cdot V+B\cdot B=\frac{1}{4}$, $V\wedge B=0$, $P=0$; (leaves four degrees of freedom)	$\mathrm{Tr}\,\psi=1$
Nilpotent	$S=0$, $P=0$, $V\cdot V+B\cdot B=0$, $V\wedge B=0$; (leaves four degrees of freedom)	$\mathrm{Tr}\,\psi=0$
Hermitian	$S^2=V\cdot V$, $B=0$, $P=0$; (leaves three degrees of freedom)	$\psi=\tilde{\psi}$

whence $S=\frac{1}{2}$ with no further restrictions. These conditions must be combined with those of table 1. A special case is $E=\frac{1}{2}(1+u)$ as is also obviously true since $E\tilde{\bar{E}}=0$. We note that Lounesto (1981) has given a method for constructing primitive idempotents in real Clifford algebras but that this does not yield *all* such idempotents associated with a given Clifford algebra. In the case of C_3 Lounesto's construction reduces to (3.12).

The non-trivial nilpotent elements in C_3 are those for which $\psi\neq 0$ and $\psi^2=0$. The necessary and sufficient condition for this is, from (3.9), $S=P=0$, $V\cdot V+B\cdot B=0$, $V\wedge B=0$. Moreover the requirement $\psi^2=2(S+P)\psi=0$, which is necessary and sufficient for an element to be a primitive nilpotent, adds nothing to these conditions and we conclude that all nilpotents in C_3 are primitive.

Finally an element is Hermitian if $\psi=\tilde{\psi}$, i.e. $B=P=0$. The necessary and sufficient condition for a primitive element to be Hermitian then follows from the combination of these restrictions with those of table 1.

4. Recovery of spin space

Having established in the last section intrinsic algebraic and tensorial conditions for a real Pauli number to be primitive, we now wish to see what these restrictions look like when a matrix representation is chosen for the basis elements of the algebra. In particular we desire to recover the column spinor as an element of a spin space (this section) and also to derive covariant tensor–spinor relations (next section).

By Wedderburn's theorem C_3, which is simple and of order 8, is isomorphic to the direct product of the algebra of 2×2 real matrices and a division algebra D of order 2. Since the centre Z of a simple algebra is a subalgebra of a division algebra in this decomposition and, for C_3, is of order 2, we see that $D=Z$. Noting further that Z is isomorphic to the complex numbers, so that we may write $e^{123}=\mathrm{i}$, it follows that C_3 is isomorphic to the algebra of 2×2 complex matrices, subject to (3.2) (and that a real Pauli number may be expressed as the sum of a complex scalar and a complex vector). Such an algebra is generated by, for example, the unit matrix I and the Pauli matrices

$$e^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad e^2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\qquad e^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\tag{4.1}$$

with $\mathrm{Tr}\,I=2$, $\mathrm{Tr}\,e^i=\mathrm{Tr}\,e^{ij}=0$, $\mathrm{Tr}\,e^{123}=2\mathrm{i}$. Thus, representing a Pauli number ψ as a complex matrix ψ^a_b, $a, b=1, 2$, we have $\mathrm{Tr}\,\psi=2(S+\mathrm{i}P_{123})$ and so $2(S+P)=(\mathrm{Tr}\,\psi)I$. The condition (3.10) now becomes in matrix language

$$\psi^2=(\mathrm{Tr}\,\psi)\psi.\tag{4.2}$$

By an easy extension of a theorem proved by Eddington (1946) it can be shown that, for $\operatorname{Tr}\psi \neq 0$, (4.2) is the necessary and sufficient condition for a square matrix to factorise into a column times a row:

$$\psi^a_b = \xi^a \eta_b \tag{4.3}$$

where, in our case, $a, b = 1, 2$ and the components of the factors are complex numbers†. When $\operatorname{Tr}\psi = 0$, which is the case for nilpotent matrices, a 2×2 matrix factorises so that the decomposition (4.3) is characteristic of any primitive element in C_3‡. As is to be expected (4.3) behaves as a generator of either a minimal left ideal or a minimal right ideal depending upon which side we multiply from. Thus, keeping ψ fixed so that it characterises a particular minimal ideal, multiplication on the left with any 2×2 matrix yields an arbitrary column times a fixed row, that is an element of the minimal left ideal generated by ψ. Effectively, then, keeping η fixed and varying ξ gives a minimal left ideal or, with ξ fixed and η varying, a minimal right ideal.

Under a rotation $e^i \to Re^iR^{-1}$, $R = \tilde{R}$, $R\tilde{R} = 1$, $R \in \mathrm{SU}(2) \subseteq C_3$ we have in (4.3) $\xi\eta \to (R\xi)(\eta R^{-1})$ i.e.

$$\xi^a \to R^a_b \xi^b \qquad R = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{4.4}$$

$$\eta_a \to \eta_b (R^{-1})^b_a \qquad R^{-1} = \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix} = (R^*)^{\mathrm{T}} \tag{4.5}$$

so that the factors transform as elements of dual vector spaces, as is indicated by the position of the indices. The factorisation condition is an intrinsic property, independent of matrix representation. (Of course we can verify the invariance of the algebraic primitivity condition directly by transforming (3.10).) We have therefore arrived at the representation of a left spinor as a column matrix and a right spinor as a row matrix.

Thus far no restrictions have been imposed on the dual spinor factors either individually or in relation to one another, i.e. we have not gone beyond the tensor conditions of table 1. Going further, let us choose $\eta = (1\ 0)$ or $\xi = \binom{1}{0}$ in the left or right cases respectively. Then

$$\psi = \begin{pmatrix} \xi^1 & 0 \\ \xi^2 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \eta_1 & \eta_2 \\ 0 & 0 \end{pmatrix}. \tag{4.6}$$

Since the tensor conditions do not distinguish between left or right ideals a restriction imposed on one of the spinor factors alone cannot have tensorial (i.e. covariant) significance. Thus in (4.6) we obtain the components ξ^a in isolation but the corresponding restriction on the tensors of ψ is non-covariant. On the other hand fixing a relationship between the factors can lead to a covariant restriction of the tensor relations. This means, however, that we can only relate quadratic combinations of column spinor components to tensors which confirms the remarks of § 1 that the sign of the spinor cannot be afforded any meaning in terms of local tensors.

The significance of fixing η to be $(1\ 0)$, say, is that this is the matrix equivalent operation to choosing a specific basis for an ideal in the algebra. This can be seen as follows. Following (3.15), a basis for the minimal left ideal generated by $E = \frac{1}{2}(1 + e^3)$

† In our case this condition is also given by $\det \psi = 0$.

‡ We may reinterpret (3.4) as proving that a 2×2 matrix is a sum of two outer products of a column with a row.

is given by $\frac{1}{2}(1+e^3)$, $\frac{1}{2}(e^1+e^{13})$, $\frac{1}{2}(e^2+e^{23})$, $\frac{1}{2}(e^{12}+e^{123})$ with real coefficients or, taking $e^{123}=\mathrm{i}$, we have for a general element of the ideal

$$\alpha^1\tfrac{1}{2}(1+e^3)+\alpha^2\tfrac{1}{2}e^1(1+e^3) \tag{4.7}$$

where α^a, $a=1,2$ are complex numbers. Similarly, an element of the minimal right ideal generated by E may be written

$$\beta_1\tfrac{1}{2}(1+e^3)+\beta_2\tfrac{1}{2}(1+e^3)e^1 \tag{4.8}$$

where β_a, $a=1,2$ are complex numbers. Taking $\frac{1}{2}(1+e^3)$ and $\frac{1}{2}e^1(1+e^3)$ or $\frac{1}{2}(1+e^3)$ and $\frac{1}{2}(1+e^3)e^1$ as the basis of a (column or row) representation space, the basis elements of the algebra have in both cases the (left or right) representation (4.1). Using (4.1) in (4.7) and (4.8) in turn we then recover the alternative forms (4.6). The same argument applies to the minimal left and right ideals generated by $E'=\frac{1}{2}(1-e^3)$. Here bases for the ideals are provided by $\frac{1}{2}(1-e^3)$, $\frac{1}{2}e^1(1-e^3)$ and $\frac{1}{2}(1-e^3)$, $\frac{1}{2}(1-e^3)e^1$ respectively, with respect to which the basis elements have a representation (different from (4.1)) which again yields the forms (4.6). Using the representation (4.1), however, the E'-generated ideals take the form which corresponds to putting $\xi=\binom{0}{1}$ or $\eta=(0\ \ 1)$ in (4.3) (with (3.13) being a sum of two column spinors). As noted above, we see that the specific choices of factors and bases are additional to (although consistent with) the tensor conditions and have no tensorial interpretation.

With respect to a particular minimal ideal (spin space) basis ψ is thus recovered as a column or row spinor with a linear vector space structure, the forms (4.6) being invariant under rotations from the left or from the right respectively. It must be noted that there are an infinite number of such spin spaces within the algebra, each one being characterised by a specific choice of generating idempotent for the ideal.

5. Recovery of known tensor–spinor relations

In order to relate the tensors of an algebraic real Pauli spinor to the components of a single column spinor in a covariant manner we require a relationship between the factors in (4.3). We shall consider here two special cases of the primitive tensor conditions which are each equivalent to such a relationship and which yield tensor representations of spinors previously proposed in the literature. The algebraic origin of these known tensor–spinor relations is hereby uncovered: nilpotency and Hermiticity. The primitive idempotents, although important to the structure of the algebra, do not appear to define on their own any spinor–tensor relations of interest.

Consider first the nilpotent restriction. According to (3.11) the unit pseudoscalar $e^{123}=\mathrm{i}$ is a duality operator so that to a bivector B there corresponds a vector W: $B=\mathrm{i}W$. Using this vector, the conditions for a nilpotent element of table 2 become

$$\psi=V+\mathrm{i}W \qquad V\cdot V=W\cdot W \qquad V\cdot W=0 \tag{5.1}$$

i.e. ψ appears as a complex vector whose nilpotency means that it is null. In matrix terms the components of the complex vector are given by

$$\tfrac{1}{2}\operatorname{Tr}(\psi e^i)=\tfrac{1}{2}\operatorname{Tr}(\eta e^i\xi)=V_i+\mathrm{i}W_i \qquad i=1,2,3. \tag{5.2}$$

Now, the nilpotent matrix ψ has zero trace: $\xi^a\eta_a=0$. This condition is satisfied by $\eta_a=\xi_a$ where $\xi_1=-\xi^2$, $\xi_2=\xi^1$ so that ξ_a is the covariant spinor equivalent to the contravariant spinor ξ^a, as is normally defined through the spin metric. Utilising the representation (4.1) in (5.2) with $\psi^a_b = \xi^a \xi_b$ yields

$$V_1 + \mathrm{i}W_1 = \tfrac{1}{2}((\xi^1)^2 - (\xi^2)^2) \qquad V_2 + \mathrm{i}W_2 = \tfrac{1}{2}\mathrm{i}((\xi^1)^2 + (\xi^2)^2) \qquad V_3 + \mathrm{i}W_3 = -\xi^1\xi^2. \tag{5.3}$$

These are the relations given by Kramers (1957) and, apart from a factor of 2, by Cartan (1966). They serve to define, in terms of a pair of equal length real orthogonal vectors (or a triad of real orthogonal vectors if we include a third vector $\mathrm{i}(V \wedge W)$), the four degrees of freedom of a two-component spinor, up to a sign. In particular, a change in the phase of ξ^a by θ appears as a rotation of the vectors V, W, as a unit, through the angle 2θ about the axis $\mathrm{i}(V \wedge W)$. Note that if ξ^a transforms as in (4.4) then ξ_a does indeed transform according to (4.5).

In the work of these authors the use of a complex vector in real space E_3 to represent a spinor appears to be a rather obscure construction. In our theory 'complex numbers', including spinor components, only occur through the identification of the unit pseudoscalar with the complex unit. The explicit use of i in (5.1) is appropriate to a restricted vector calculus in which one cannot add tensors of different rank (vector plus bivector) and it represents in the work of Cartan and Kramers a tacit introduction of e^{123}. The construction of these writers is just a way of formulating the theory of primitive nilpotents in C_3 where $e^{123} = \mathrm{i}$.

Turn now to the primitive Hermitian elements (table 2). We may choose in this case $\eta_a = \xi^{a*}$ (which transforms as in (4.5)) and, with $\psi = S + V$, $S = \frac{1}{2}\mathrm{Tr}\,\psi$, $V^i = \frac{1}{2}\mathrm{Tr}(\psi e^i)$:

$$\begin{aligned} S &= \tfrac{1}{2}(|\xi^1|^2 + |\xi^2|^2) \qquad & V^1 &= \tfrac{1}{2}(\xi^{1*}\xi^2 + \xi^{2*}\xi^1) \\ V^2 &= \tfrac{1}{2}\mathrm{i}(\xi^1\xi^{2*} - \xi^2\xi^{1*}) \qquad & V^3 &= \tfrac{1}{2}(|\xi^1|^2 - |\xi^2|^2). \end{aligned} \tag{5.4}$$

As noted by Takabayasi (1955) in this interpretation the tensors serve to define only three of the degrees of freedom of the spinor components, the phase of ξ^a not being reflected by (5.4) (since (5.4) depends only on the difference of the phases of the components of ξ^a).

Specialising still further, by normalising with $\mathrm{Tr}\,\psi = 1$, we obtain the primitive Hermitian idempotent elements—it is these which are used in the standard density matrix treatment of a non-relativistic spin-$\frac{1}{2}$ particle (Schiff 1968, p 381). Such elements have the form $\frac{1}{2}(1 + u)$, u a unit polarisation vector, as in (3.12), and these primitive idempotents represent pure states. The density matrix is a Hermitian element of trace 1 which decomposes into a sum of primitive elements (pure states).

6. Summary

The aim has been to systematise the various non-covariant and covariant relations between tensors and spinors in C_3 by showing that they are particular cases of a single algebraic requirement. We have found the covariant subsidiary conditions on the tensors of a real Pauli number for this number to be primitive and have the form (4.3). Restricting the tensor conditions in a non-covariant way (as in (3.15)) yields in a linear fashion the group-theoretical components of a spinor (4.6) whereas restricting them in a covariant way (as in table 2) yields the components of a spinor but only quadratic combinations thereof. In the latter case we have shown that the nilpotent restriction gives a set of tensors which carry the same information as a column spinor, apart from a sign, and the Hermitian restriction the same information apart from a phase.

It is an outstanding problem to extend this work so as to state the condition for primitivity, as a restriction formulated in terms of just one algebraic element, for any Clifford algebra. This is readily done, however, in the case of the Dirac algebra, as will be reported elsewhere.

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